Computer Architecture
Digital Circuits

http://d3s.mff.cuni.cz/teaching/nswi143

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CHARLES UNIVERSITY IN PRAGUE

faculty of mathematics and physics

Digital computer

- **Two voltage levels of interest**
  - **High level**
    - Logical one (signal *high/true/asserted)*
  - **Low level**
    - Logical zero (signal *low/false/deasserted)*
  - Logical values are complementary and inverse of each other
    - Unlike the voltage levels representing logical ones and zeros

Logic blocks

- **Combinational**
  - No memory → no internal state
  - Output depends only on current input
  - Represents logical functions

- **Sequential**
  - Has memory → has internal state
  - Output depends on input and internal state
  - Captures sequence of steps

Logic functions and truth tables

- **Logic function (also Boolean function)**
  - Output value is a function of input values
    - f: $\mathbf{B}^k \rightarrow \mathbf{B}$, where $\mathbf{B} = \{ 0, 1 \}$ and $k \in \mathbf{N}$ is arity
  - Truth table
    - Function defined by enumerating the output for each combination of inputs (a table 2^k rows for k inputs)

Inputs		**Output**
a	***b***	***f*(*a*, *b*)**
0	0	0
0	1	1
1	0	1
1	1	1

Boolean algebra

- **Logic functions expressed as equations**
  - Variables hold values from **B** = {0, 1}
  - Basic operators – primitive logic functions
    - **Logical inversion** (NOT): $\overline{x}$, $\neg x$, $!x$
    - **Logical product, conjunction** (AND): $x \cdot y$, $x \land y$, $x \mathbin{\&\&} y$
    - **Logical sum, disjunction** (OR): $x + y$, $x \lor y$, $x \mathbin{||} y$
  - Additional operators (16 for 2 variables)
    - NAND, NOR, XOR etc.

Logic operators

Inputs		Basic operators			Universal operators	
a	*b*	NOT *a*	*a* AND *b*	a OR *b*	*a* NAND *b*	*a* NOR *b*
		¬	∧	∨	↑	↓
0	0	1	0	0	1	1
0	1	1	0	1	1	0
1	0	0	0	1	1	0
1	1	0	1	1	0	0

Inputs		Other operators						
a	*b*	*a* XOR *b*	*a* XNOR *b*					
		⊕	↔	→	←	…	⊥	⊤
0	0	0	1	1	1	...	0	1
0	1	1	0	1	0	...	0	1
1	0	1	0	0	1	...	0	1
1	1	0	1	1	1	...	0	1

Boolean algebra laws

- **Idempotency:** $a + a = a$, $a \cdot a = a$
- **Computativity:** $a + b = b + a$, $a \cdot b = b \cdot a$
- **Associativity:** $a + (b + c) = (a + b) + c$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- **Absorption:** $a \cdot (a + b) = a$, $a + (a \cdot b) = a$
- **Distributivity:** $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$, $a + (b \cdot c) = (a + b) \cdot (a + c)$
- **Neutrality of 0 and 1:** $a + 0 = a$, $a \cdot 1 = a$
- **Aggressivity of 0 and 1:** $a + 1 = 1$, $a \cdot 0 = 0$
- **Complementarity:** $a + \neg a = 1$, $a \cdot \neg a = 0$
- **Absorption of negation:** $a \cdot (\neg a + b) = a \cdot b$, $a + (\neg a \cdot b) = a + b$
- **De Morgan’s laws:** $\neg(a + b) = \neg a \cdot \neg b$, $\neg(a \cdot b) = \neg a + \neg b$
- **Double negation:** $\neg(\neg a) = a$

Intermezzo: CPU logical operations (1)

- **Logic functions extended to operate on (finite) sequences of bits**
  - Word = finite sequence of bits
  - Word length = number of bits in the sequence
  - Output of a logic operation a is function of input values
    - $f: (\mathbf{B}^n)^k \rightarrow \mathbf{B}^n$, where $\mathbf{B} = \{0, 1\}$, $k \in \mathbf{N}$ is arity and $n \in \mathbf{N}$ is word length

Intermezzo: CPU logical operations (2)

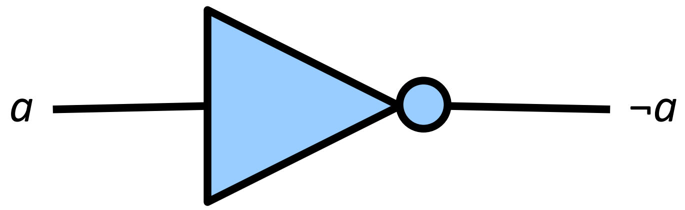
- **(Bitwise) logical product/sum/inversion**
  - Operators **&**, **|**, **~** etc. in C-like languages
  - Primitive logic function applied to individual bits of the input words, result stored to individual bits of the output word
  - Allow isolating (AND), zeroing (AND, NOR), setting (OR), inverting (XOR) selected bits, or inverting all bits (NOT), of the input word

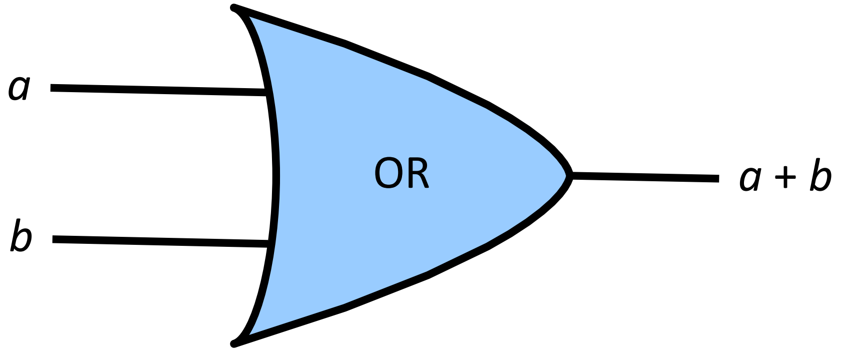
Intermezzo: CPU logical operations (3)

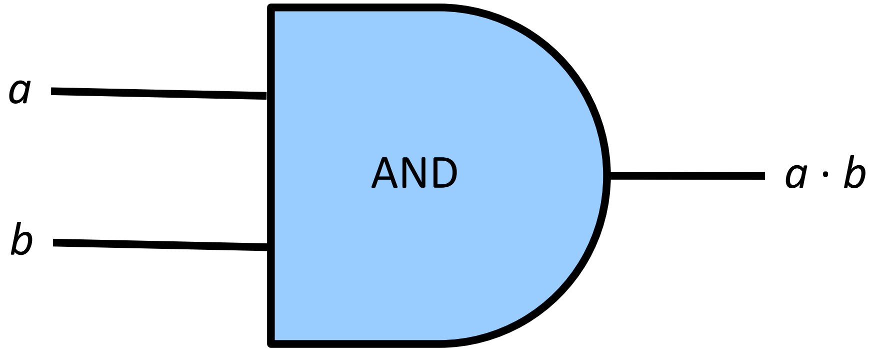
- **Logical shifts (left and right)**
  - Operators **<<** and **>>** in C-like languages
  - Shifts bits in a words ***i*** positions to the left or right
    - “Vacated” bits are replaced with 0
  - For binary natural numbers
    - Shift by ***i*** bits to the left → multiplying by $\mathbf{2}^{i}$
    - Shift by ***i*** bits to the right → dividing by $\mathbf{2}^{i}$

Logic gates (1)

- **Physical implementation basic logic functions**
  - Basic gates: NOT, OR, AND

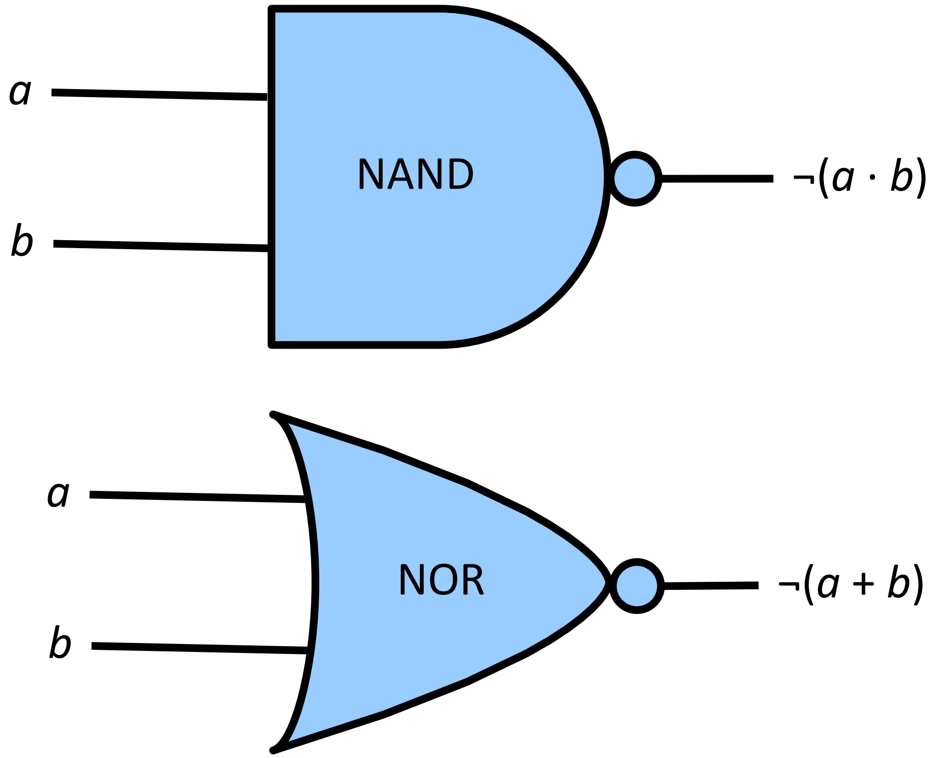


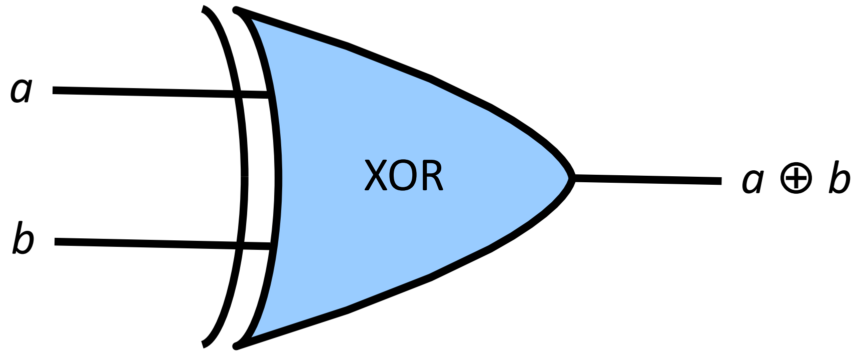




Logic gates (2)

- **Physical implementation of logic operators**
  - Inverting gates: NAND, NOR
  - Less common gates: XOR



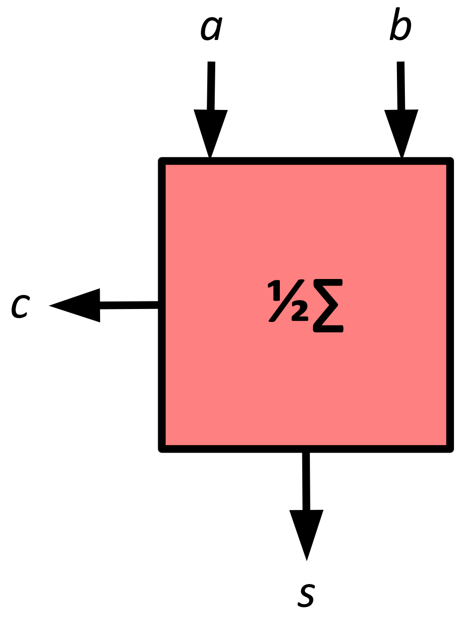


Combinational logic circuits

- **Implementation of more complex logic functions**
  - Combines multiple logic operators
    - Logic signals correspond to variables
    - Logic gates correspond to primitive operators
  - Most commonly NAND or NOR gates
    - Sufficient for expressing any logic function
- **Logic block**
  - Abstracts away from internal structure of a circuit
  - Provides functional building blocks

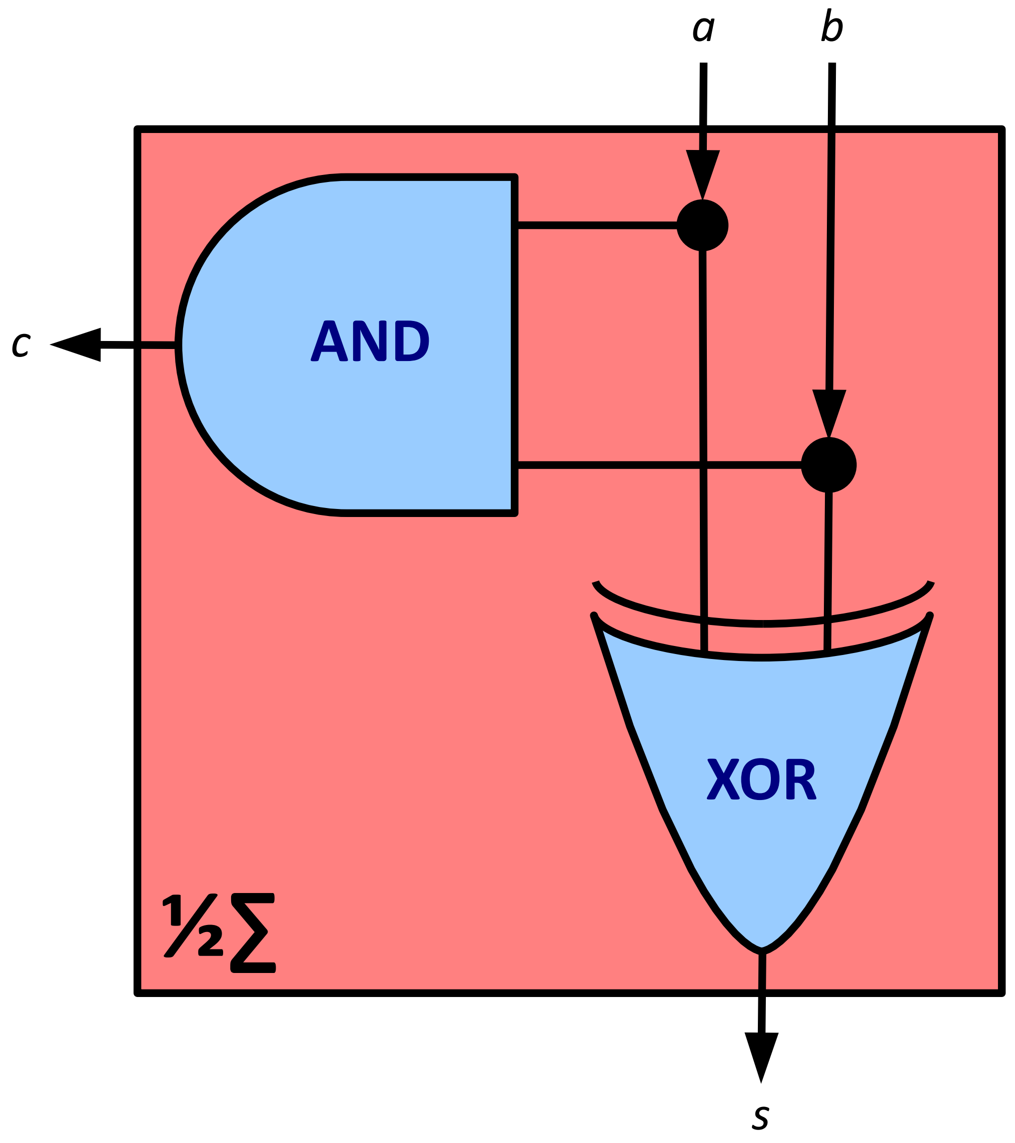
Logic blocks: binary (half) adder

- **Adds two 1-bit numbers**
  - The simplest case
  - **Input:**
    - operand ***a***
    - operand ***b***
  - **Output:**
    - sum ***s***
    - carry ***c***
  - **Function:**
    - $s = a \cdot \neg b + \neg a \cdot b = a \text{ XOR } b$
    - $c = a \cdot b = a \text{ AND } b$



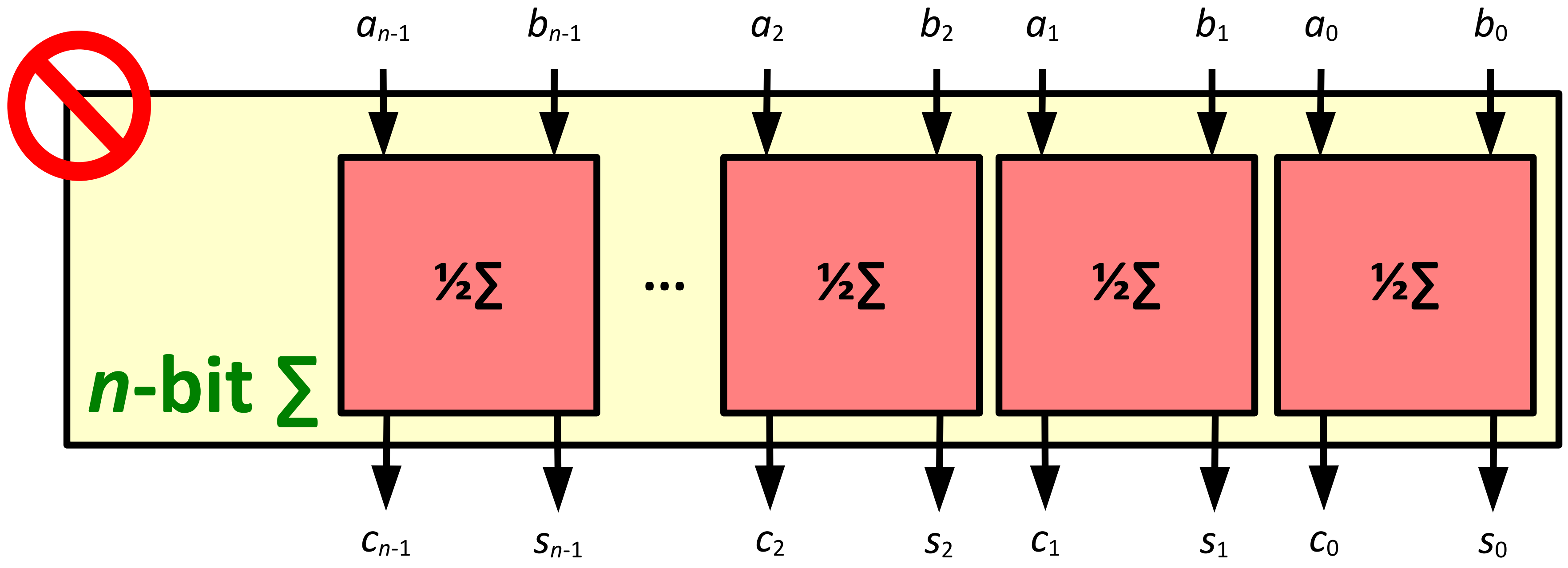
Inputs		Outputs	
a	***b***	***c***	***s***
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Logic blocks: binary (half) adder (2)



Logic blocks: binary adder (3)

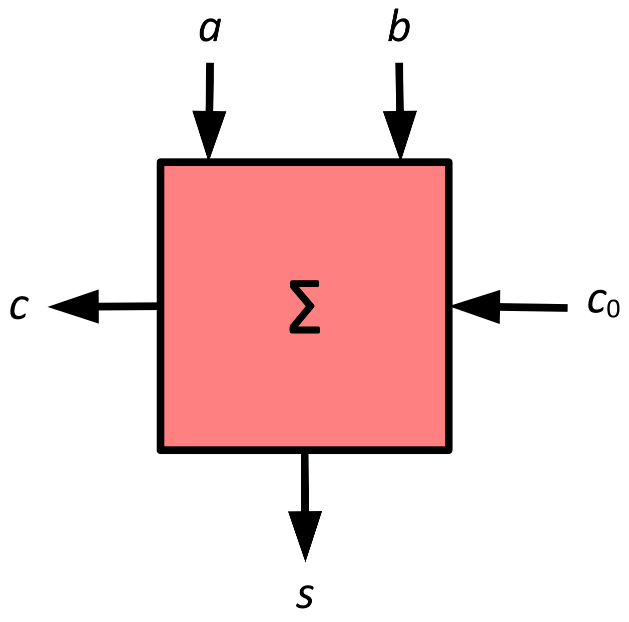
- **Adding *n*-bit numbers**
  - Merge ***n*** ½-adders for individual bits?



- ½-adder cannot propagate carry from previous additions (not enough inputs)

Logic blocks: binary adder (4)

- **Full adder**
  - Adds two 1-bit numbers taking into account carry from previous addition
  - **Input:**
    - operand ***a***
    - operand ***b***
    - carry ***c_0***
  - **Output:**
    - sum ***s***
    - carry ***c***

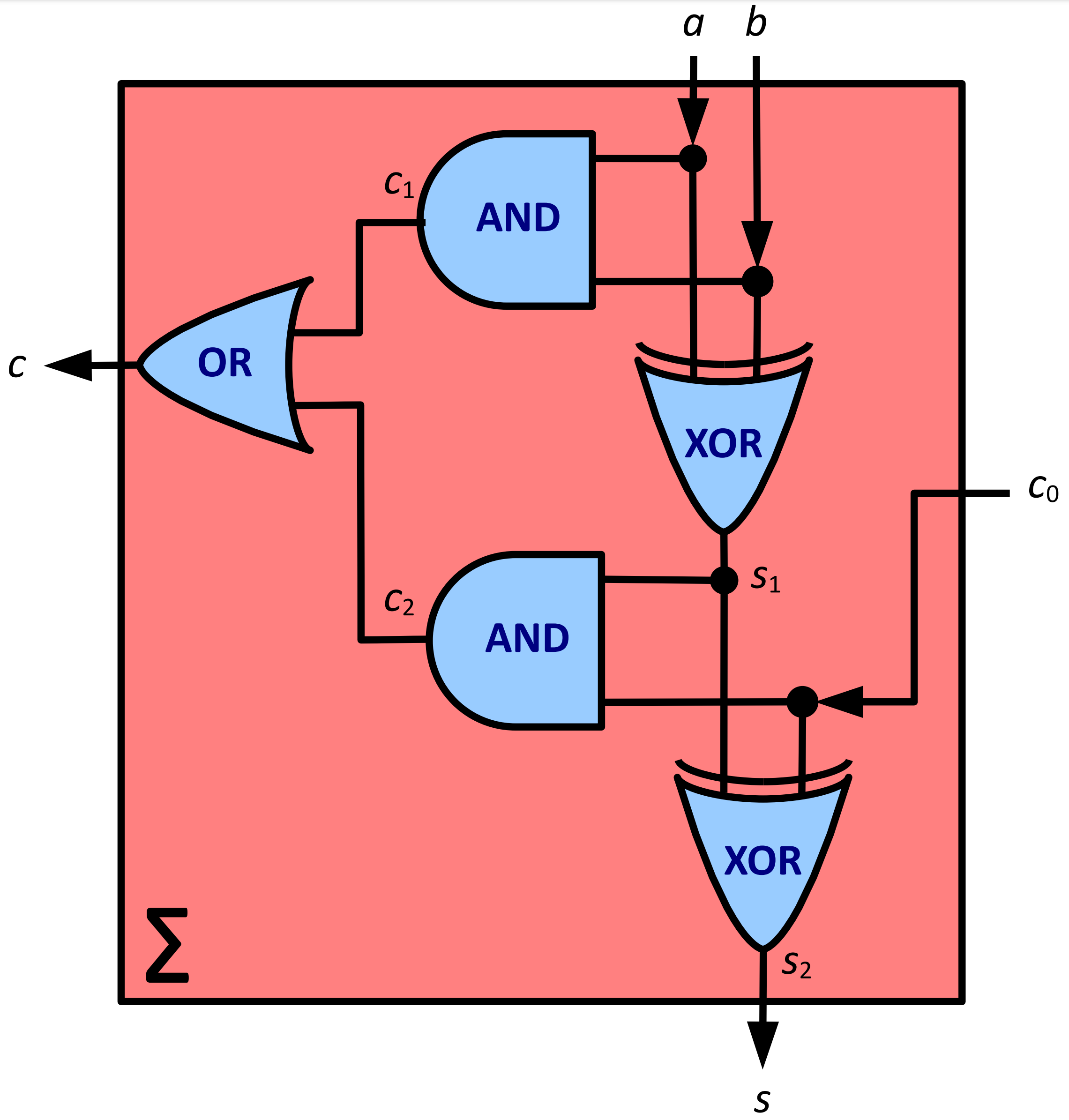


Logic blocks: binary adder (5)

- **Full adder**
  - Adds two 1-bit numbers taking into account carry from previous addition
  - **Inputs:** operand ***a***, operand ***b***, carry ***c₀***
  - **Outputs:** sum ***s***, carry ***c***
    - $s = \neg c_0 \cdot (a \cdot \neg b + \neg a \cdot b) + c_0 \cdot (a \cdot b + \neg a \cdot \neg b)$
      $s = \ldots$
      $s = c_0$ XOR (a XOR b)
    - $c = a \cdot b + c_0 \cdot (a \cdot \neg b + \neg a \cdot b)$
      $c = (a$ AND $b)$ OR $(c_0$ AND $(a$ XOR $b))$

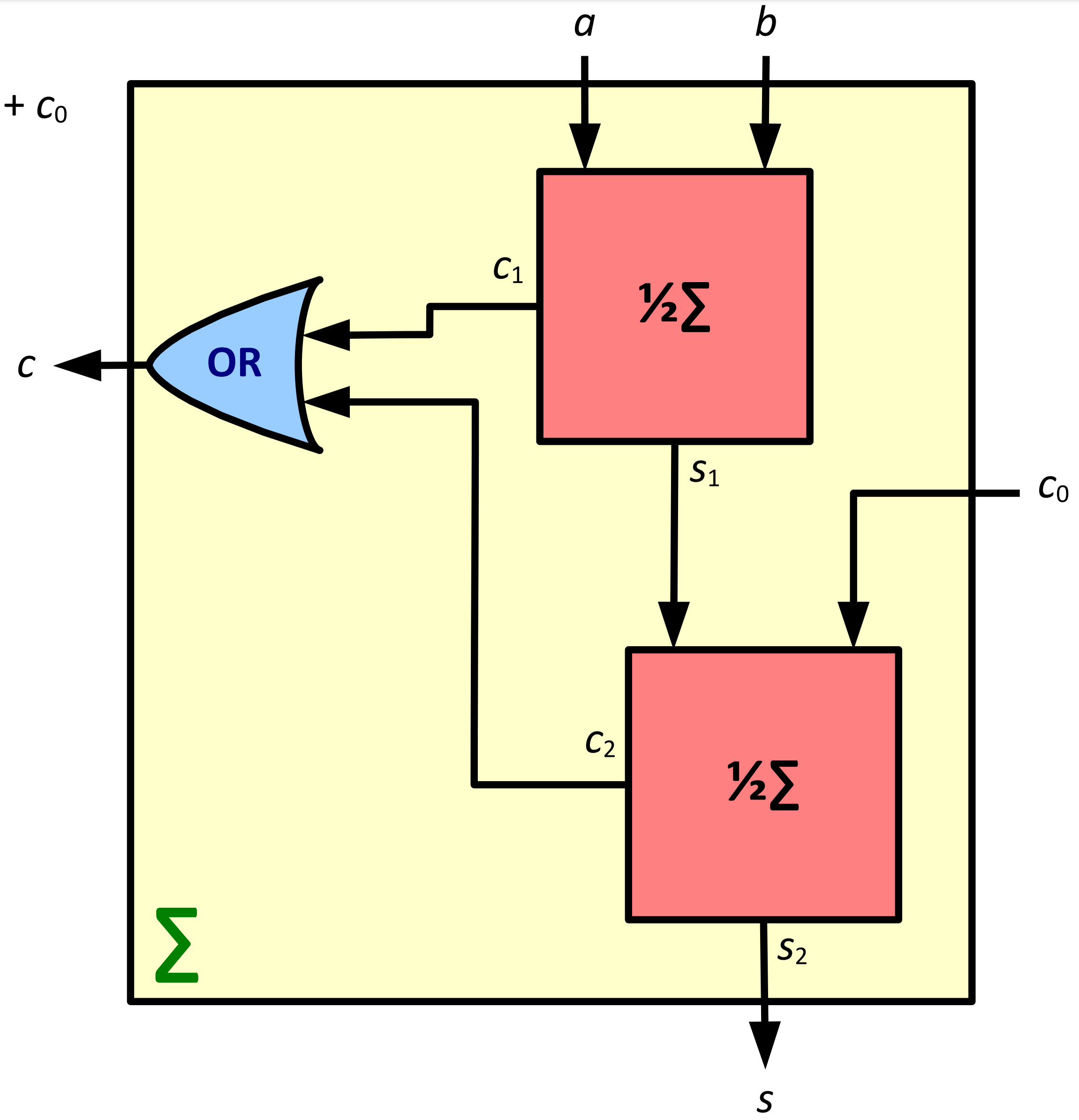
c_0	a	b	c	s
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Logic blocks: binary adder (6)

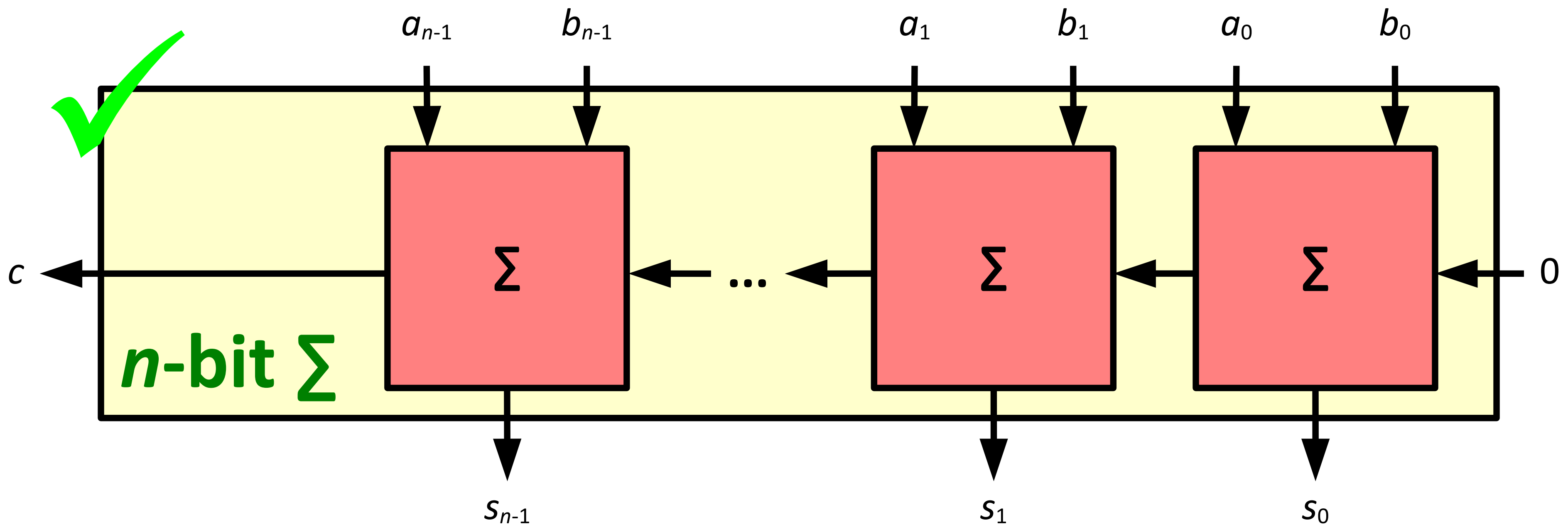


Logic blocks: binary adder (7)

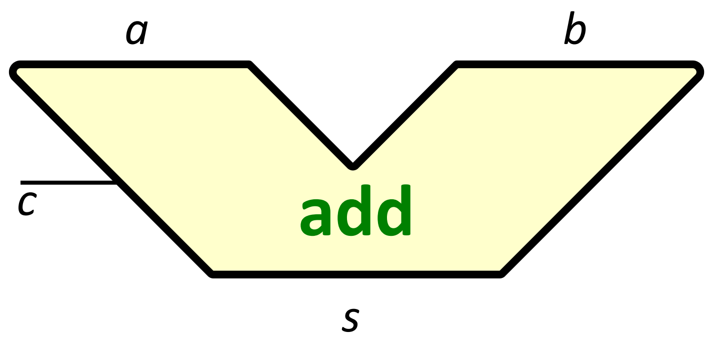
Abstraction:
$a + b + c_0 = (a + b) + c_0$



Logic blocks: binary adder (8)



Function block:

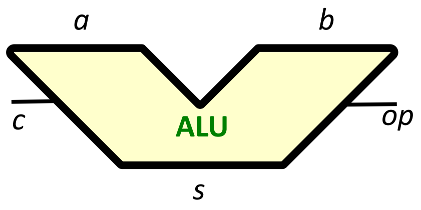


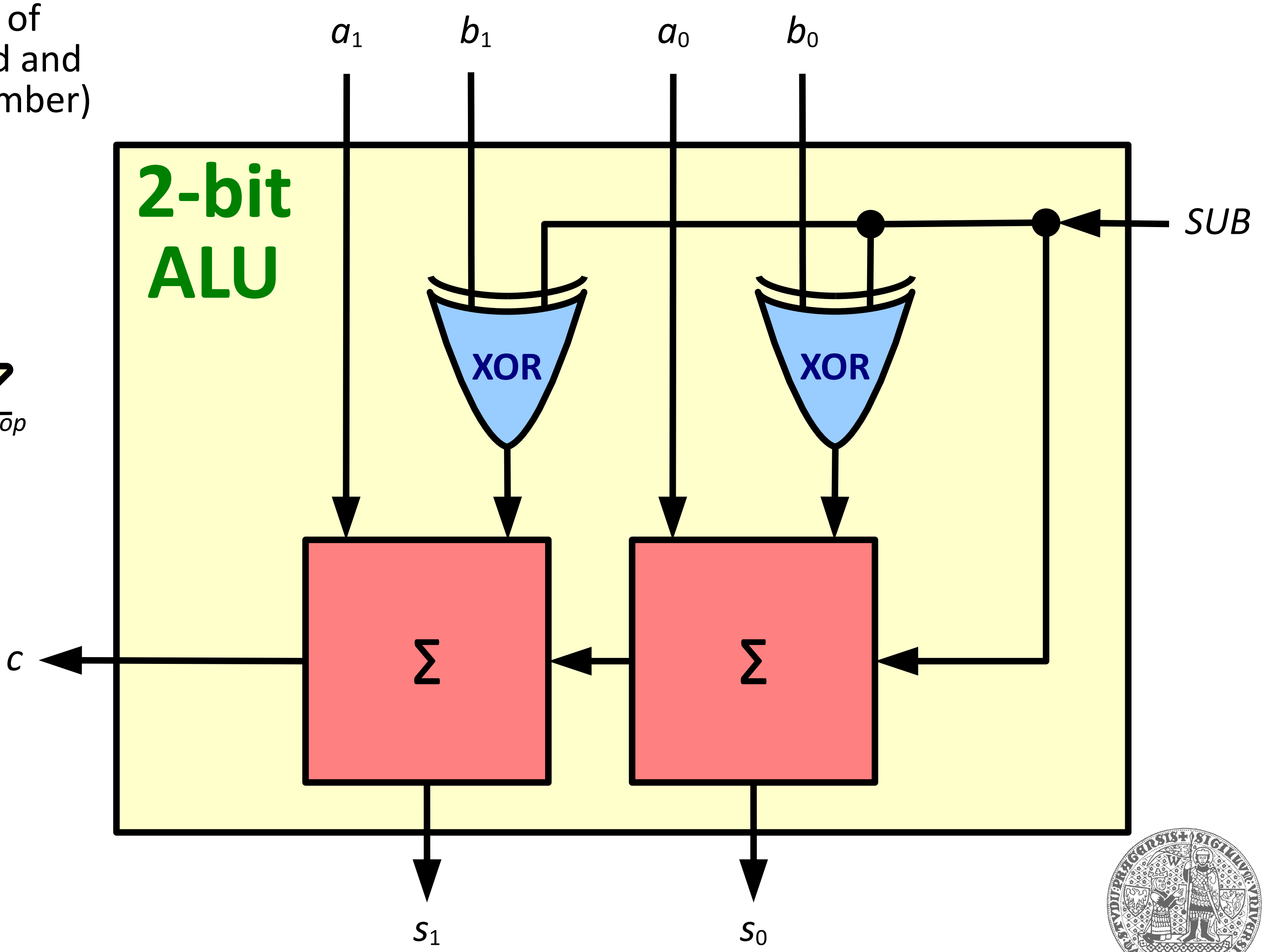
Logic block for subtraction

- Taking advantage of 2’s complement
  - Basic building block: adder
  - Use XOR gate as a controlled inverter
  - **Example:** 2-bit ALU supporting addition and subtraction
    - **Data input:** operand bits a_1a_0, operand bits b_1b_0
    - **Control input:** signal **SUB** to determine operation
      - $SUB = 0 \rightarrow$ addition
      - $SUB = 1 \rightarrow$ subtraction
    - **Output:** sum/difference bits s_1s_0, carry c

2-bit ALU for adding/subtracting

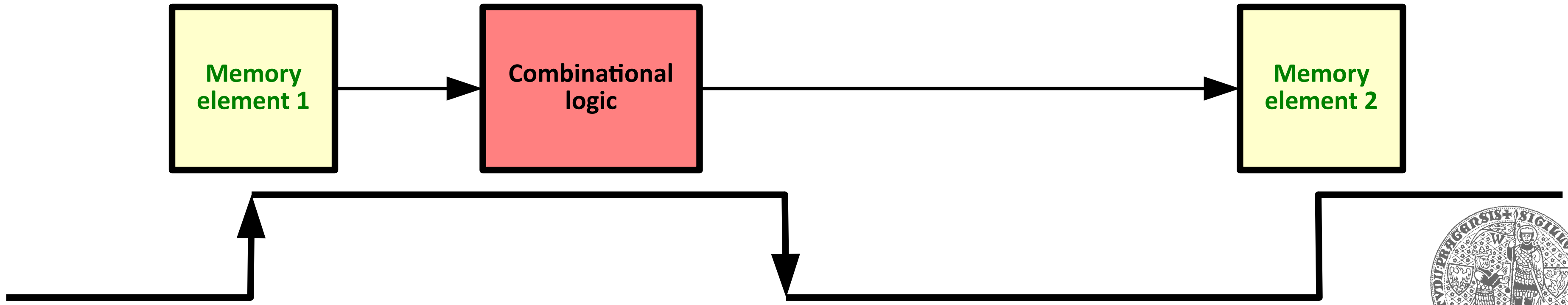
SUB = 1 inverts bits of the second operand and adds 1 (negates number)





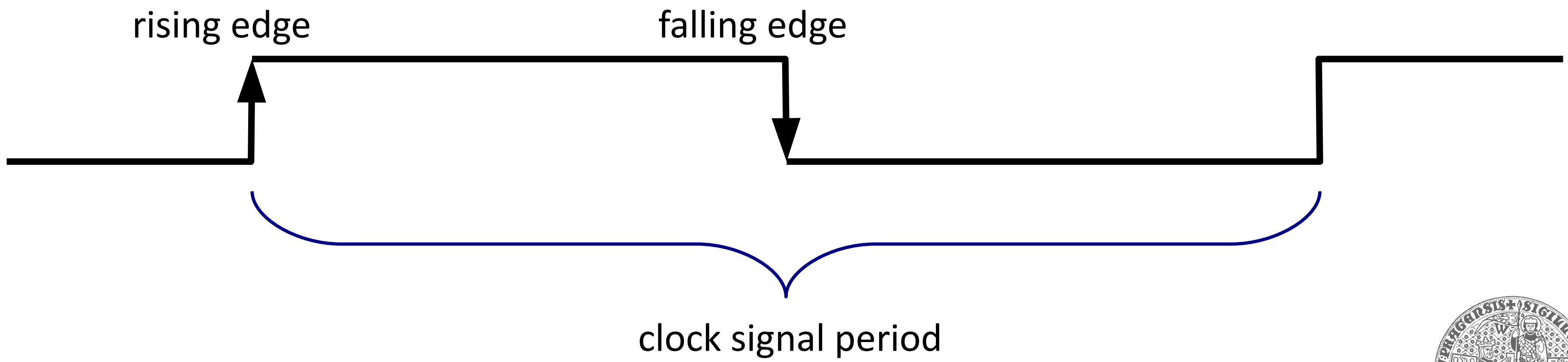
Sequential logic

- **Combinational logic + memory elements**
  - Memory elements keep internal state
  - Inputs and the contents of memory (internal state) determines outputs and next internal state
    - Synchronous vs. asynchronous sequential circuits
      - Determines how and when state changes
      - Need to ensure stable inputs (inputs don't change)



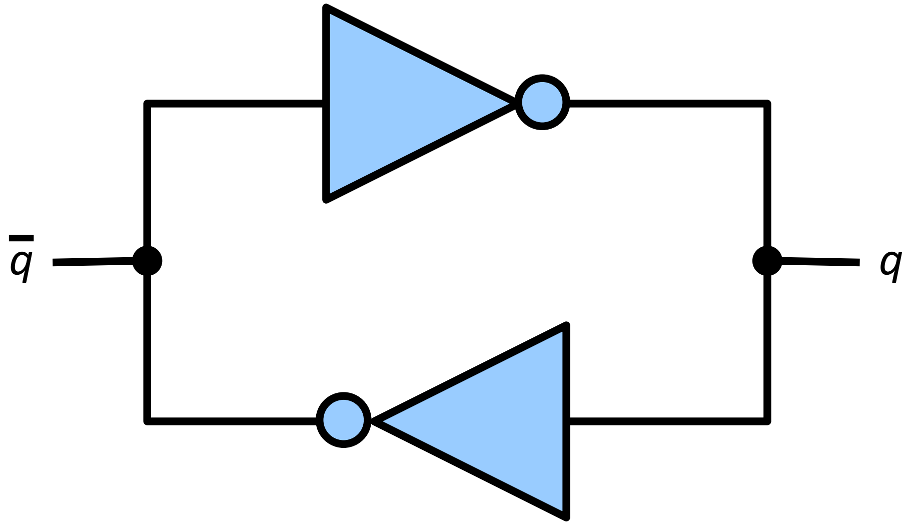
Synchronous sequential circuits

- Clock signal to synchronize state changes
  - Change state during one clock cycle
    - Inputs of combinational logic does not change while it is being read
    - Writing of values from outputs to memory elements happens with rising/falling edge of the clock signal

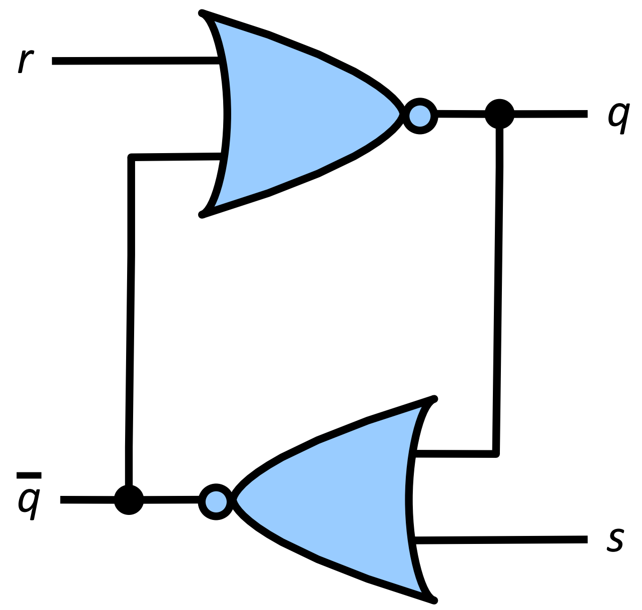


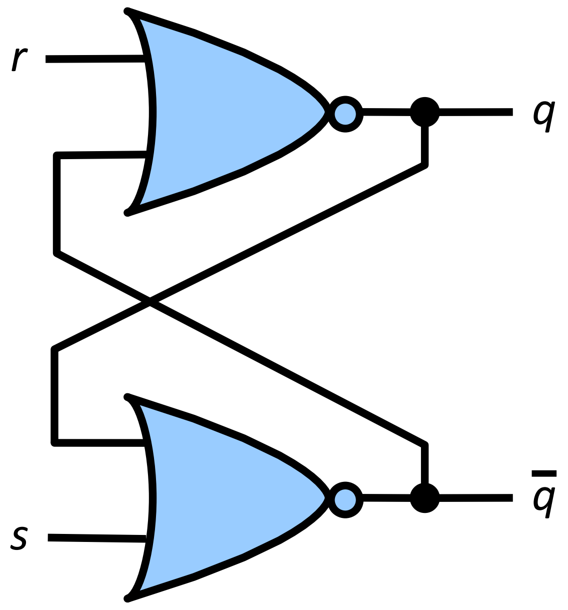
Memory elements

- **Pair of inverters in a feedback loop**
  - Asynchronous circuit with two stable states
    - Allows "storing" 1 bit of information
    - Need to be able to control the state…
      - We need a gate that can pass the signal unchanged, but allows forcing an output value when required
  - Basic building block for memory elements



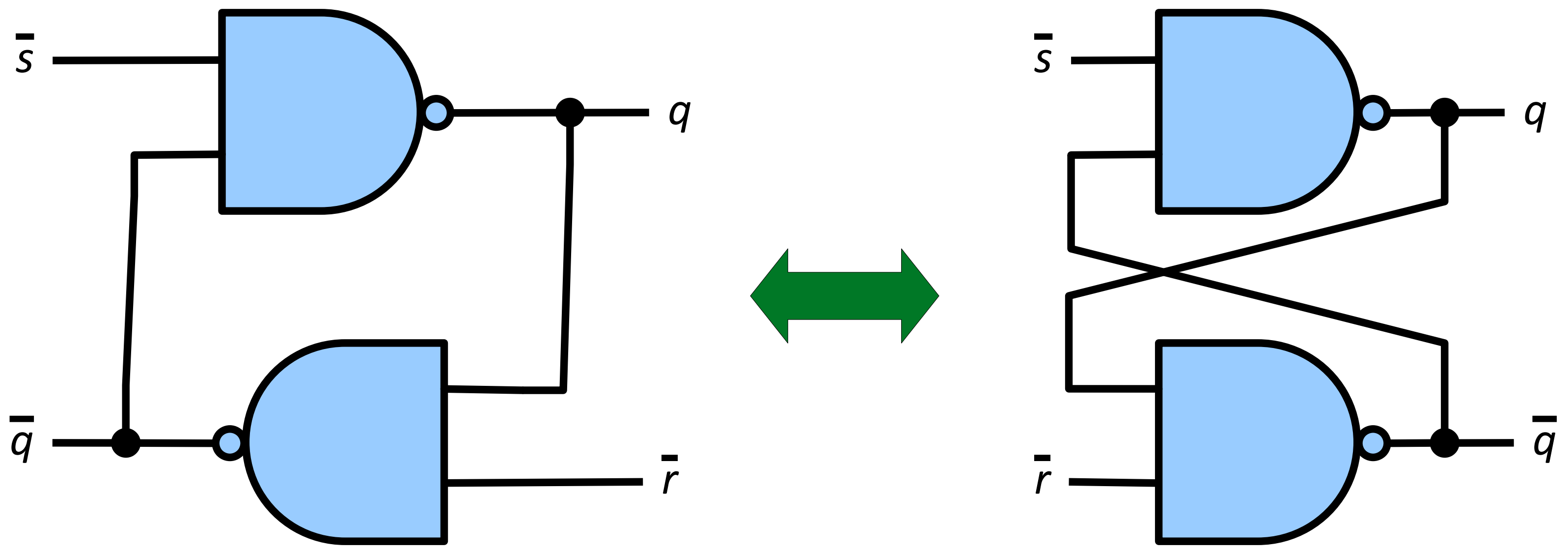
Set-Reset (R-S) latch, NOR-based





Inputs		Outputs	
r	***s***	q_n	$\bar{q}_n$
0	0	q_{n-1}	$\neg q_{n-1}$
0	1	1	0
1	0	0	1
1	1	?	?

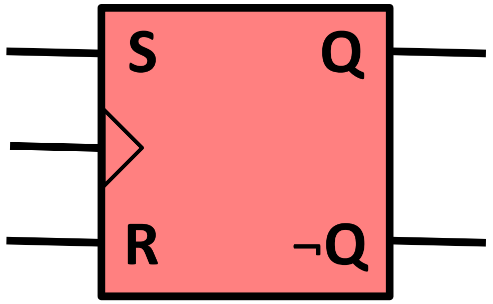
Set-Reset (R-S) latch, NAND-based



Inputs		Outputs	
$\bar{r}$	$\bar{s}$	q_n	$\bar{q}_n$
0	0	?	?
0	1	0	1
1	0	1	0
1	1	q_{n-1}	$\neg q_{n-1}$

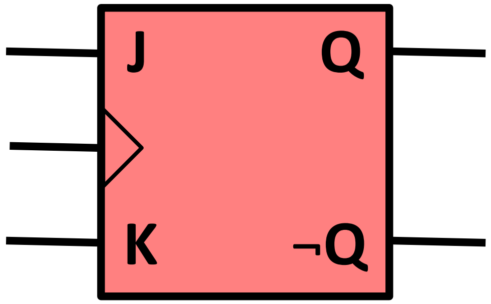
Other flip-flops

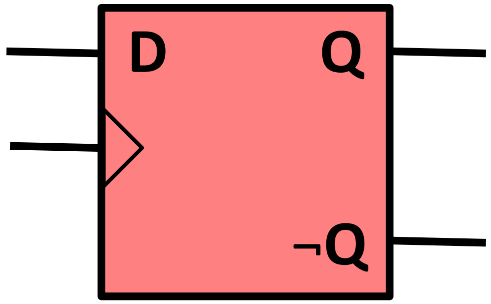
- **Derived from R-S**
  - *Clocked R-S latch*
    - Synchronous R-S latch variant
    - Reacts to ***R*** or ***S*** signals while the clock signal is high
  - *R-S master/slave* (*R-S flip-flop*)
    - Two clocked R-S latches (in series) with complementary clock signal
    - Reacts to ***R*** or ***S*** signals only on rising/falling edge of the clock signal

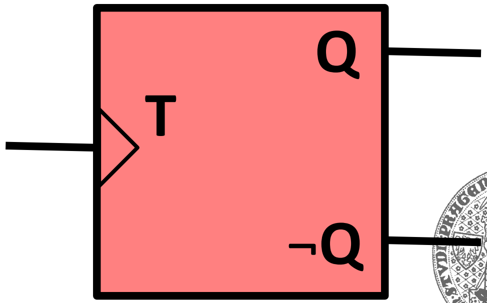


Other flip-flops (2)

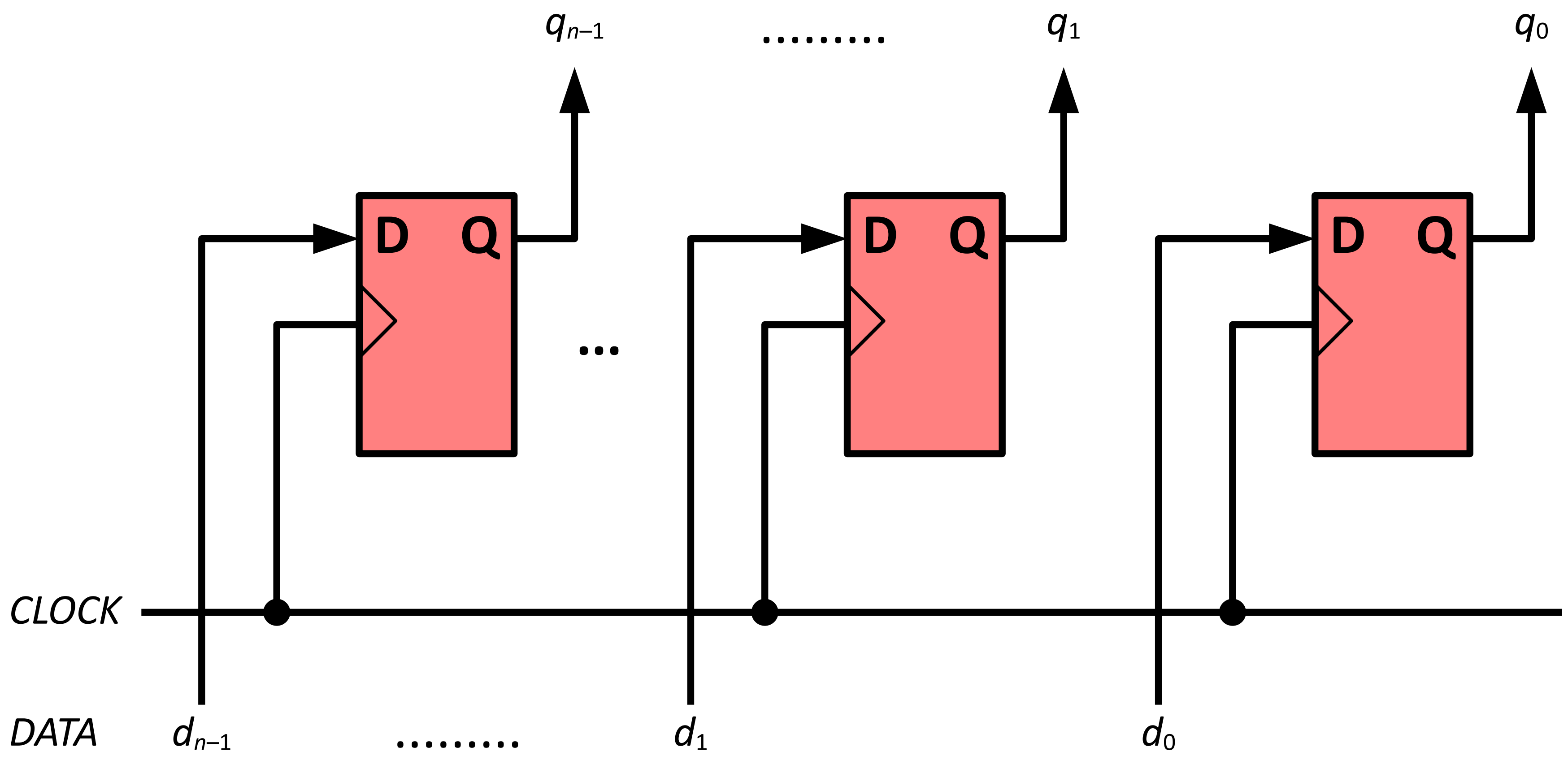
- **Derived from R-S**
  - *J-K master/slave* (*J-K flip-flop*)
    - Extends R-S (J = S, K = R), inverts state when J = K = 1
  - *Clocked D latch, D flip-flop*
    - Value determines by single input
  - *T flip-flop*
    - Allows dividing clock signal frequency



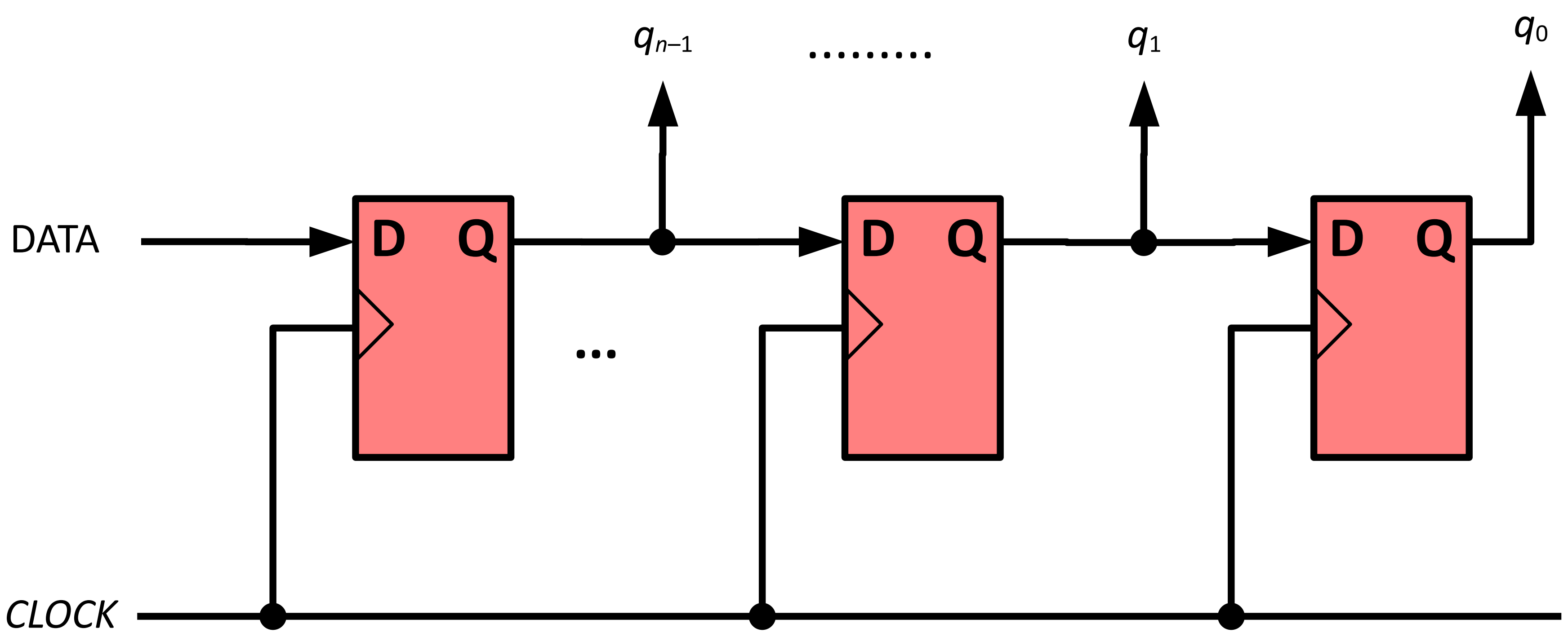




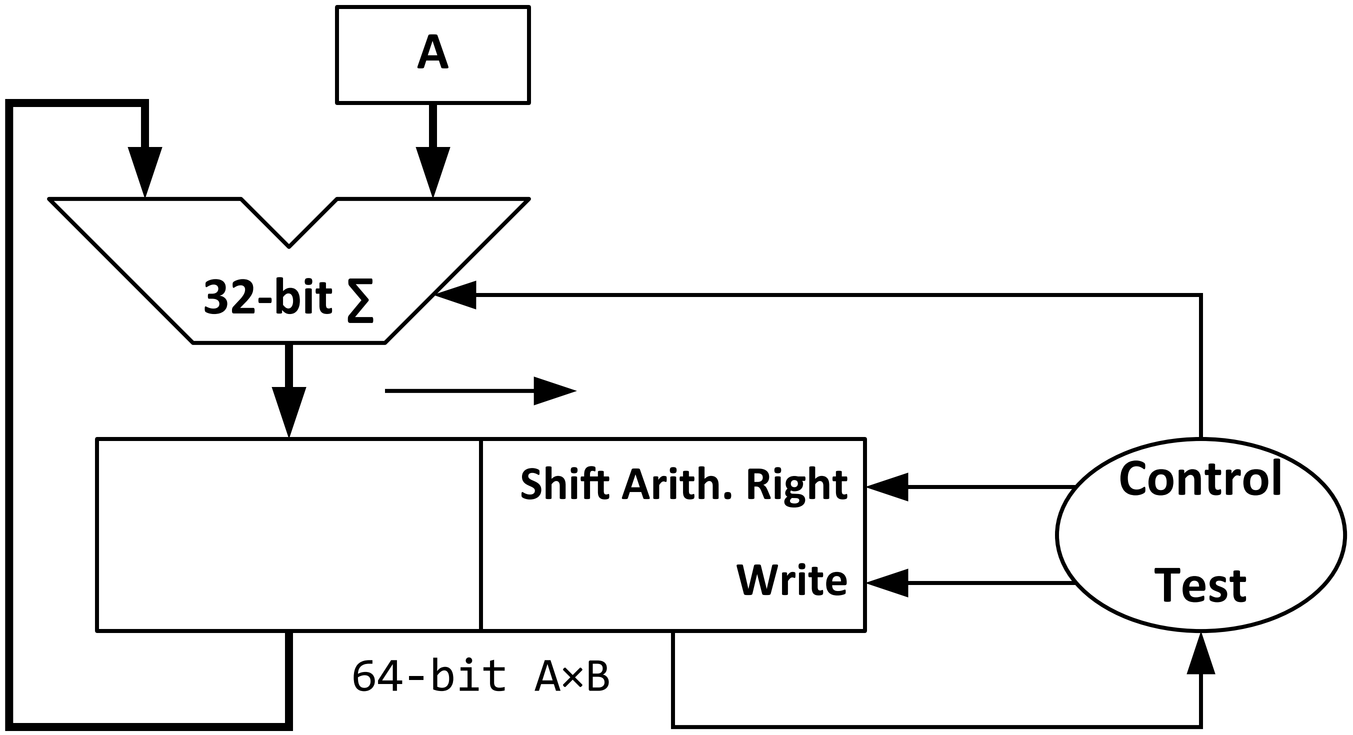
Data register made of flip-flops



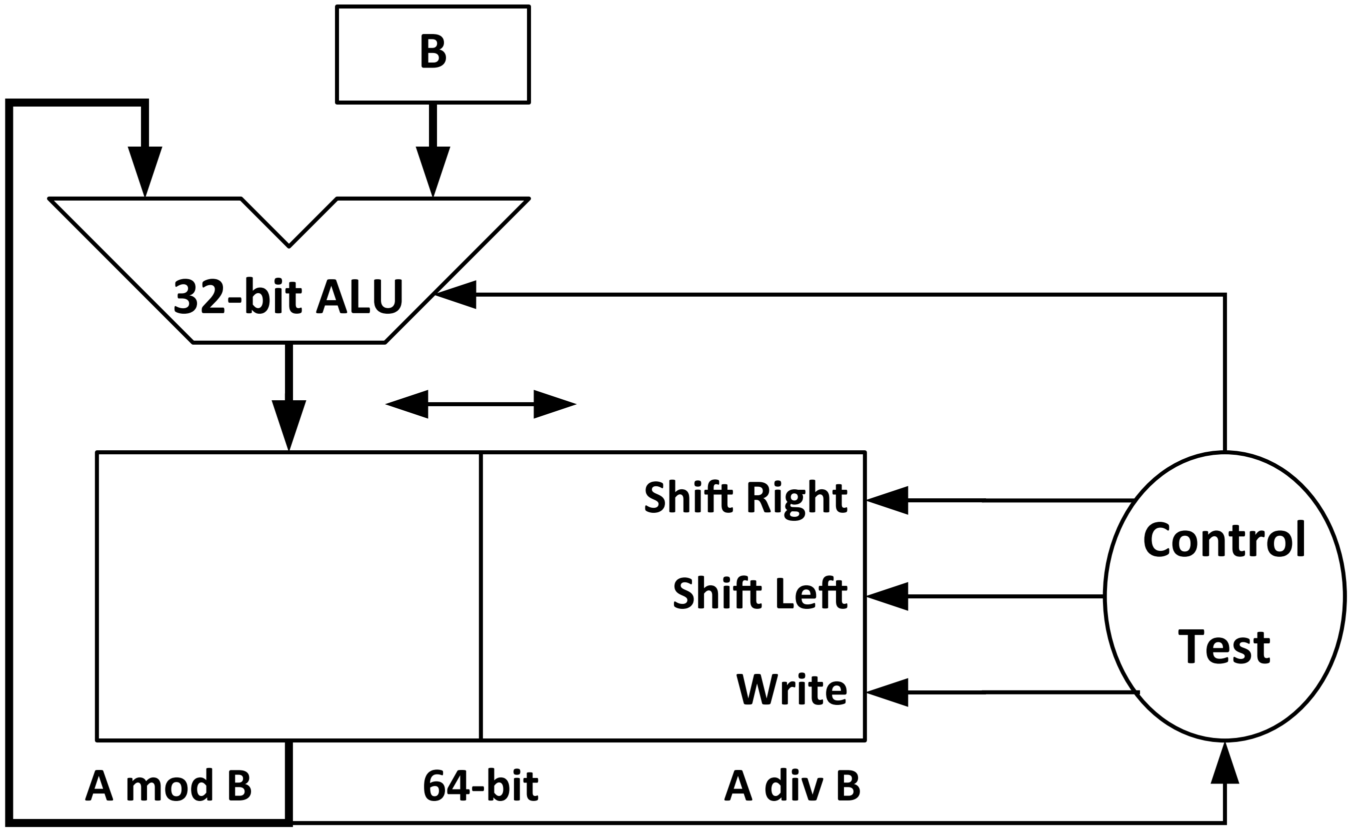
Shift register made of flip-flops



32-bit sequential multiplier



32-bit sequential divider


B
32-bit ALU
Shift Right
Shift Left
Write
Control
Test
A mod B
64-bit
A div B